Exercise 25

Use a calculator to evaluate the line integral correct to four decimal places.

$$\int_C xy \arctan z \, ds, \quad \text{where } C \text{ has parametric equations } x = t^2,\ y = t^3,\ z = \sqrt{t}, \quad 1 \le t \le 2$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}
\int_C xy \arctan z \, ds &= \int_1^2 x(t)y(t) \arctan z(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \\
&= \int_1^2 (t^2)(t^3) \left(\arctan \sqrt{t}\right) \sqrt{(2t)^2 + (3t^2)^2 + \left(\frac{1}{2} t^{-1/2}\right)^2} \, dt \\
&= \int_1^2 \left(t^5 \arctan \sqrt{t}\right) \sqrt{4t^2 + 9t^4 + \frac{1}{4t}} \, dt \\
&= \int_1^2 \frac{t^4 \arctan \sqrt{t}}{4} \sqrt{64t^4 + 144t^6 + 4t} \, dt.
\end{aligned}$$

Let $f(t)$ represent the integrand.

$$\int_C xy \arctan z \, ds = \int_1^2 f(t) \, dt$$

Use Simpson's rule with $n = 10$.

$$\begin{aligned}
\int_C xy \arctan z \, ds &\approx \frac{\Delta t}{3} [f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4) + 4f(t_5) \\
&\qquad + 2f(t_6) + 4f(t_7) + 2f(t_8) + 4f(t_9) + f(t_{10})] \\
&\approx \frac{2-1}{3(10)} \left[f(1) + 4f\left(\frac{11}{10}\right) + 2f\left(\frac{6}{5}\right) + 4f\left(\frac{13}{10}\right) + 2f\left(\frac{7}{5}\right) + 4f\left(\frac{3}{2}\right) \right. \\
&\qquad \left. + 2f\left(\frac{8}{5}\right) + 4f\left(\frac{17}{10}\right) + 2f\left(\frac{9}{5}\right) + 4f\left(\frac{19}{10}\right) + f(2) \right] \\
&\approx \frac{2-1}{3(10)} (2844.86) \\
&\approx 94.8288
\end{aligned}$$